On Triangles with Sides That Form an Arithmetic Progression

Yu.N. Maltsev, A.S. Monastyreva

Altai State Pedagogical University (Barnaul, Russia)

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Properties of triangles such that the squares of their sides form an arithmetic progression were studied in 2018. In this paper, triangles with sides that form an arithmetic progression are described. Let a, b, c be sides of an arbitrary triangle ABC. If sides b, a, c of the triangle ABC form an arithmetic progression then, for example, the equality \( a = (b + c)/2 \) (b < a < c) holds. The class of triangles for which equality \( a = (b + c)/2 \) is greater than the class of triangles for which \( b < a < c \) is described. In this paper, we study the properties of triangles for which this equality holds. Thus, triangles with sides that form an arithmetic progression are described with the help of the parameters \( p, R, r \). Classes of rectangular triangles, triangles with angle 30\(^\circ\), triangles with angle 60\(^\circ\), triangles with angle 120\(^\circ\) are studied and described.

**Key words**: triangle, circumradius, inradius, semiperimeter, arithmetic progression.

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1. Introduction. Let \( R \) and \( r \) be the circumradius and the inradius of an arbitrary triangle \( ABC \). Also, let \( AB = c, AC = b, BC = a, \angle A = \alpha, \angle B = \beta, \angle C = \gamma, p = \frac{a + b + c}{2} \).

In [1], the authors studied properties of triangles such that the squares of its sides form an arithmetic progression. Also, description of such triangles associated with its remarkable points is given. It is natural to describe triangles such that its sides form an arithmetic progression. In this case, for examples, the equality \( a = \frac{b + c}{2} \) (b < a < c) holds. The class of triangles for which \( a = \frac{b + c}{2} \) is greater than the class of triangles for which \( b, a, c \) form an arithmetic progression. Really, the class of triangles for which \( a = \frac{b + c}{2} \) also contains all equilateral triangles. Further, this class contains rectangular triangle with sides 3, 4, 5. In [2, Ex. 352], [3, Ex. 286, 287, 321], [4, Ex. 11.55, 12.23], [5, p. 89, Ex. 38], and [6, Ex. 88], some properties and characterizations of such triangles are given.

In this paper, study of this triangles is continued and the following theorems are proved [7].
Theorem 1. For arbitrary triangle ABC, the following conditions are equivalent:
1. \( a = \frac{b + c}{2} \);
2. \( p^2 = 18Rr - 9r^2 \);
3. the sides of the triangle ABC are equal to \( \frac{2p}{3}\sqrt{2r(R - 2r)}, \frac{2p}{3}\sqrt{2r(R - 2r)} \).

Besides this, \( \alpha \leq 60^\circ \).

Theorem 2. Let \( R, r \) be arbitrary positive numbers such that \( R \geq 2r \) and \( p = \sqrt{18Rr - 9r^2} \). Then there exists a unique triangle ABC such that \( R, r, p \) are the circumradius, the inradius, the semiperimeter of the triangle ABC respectively.

Theorem 3. (1) Let \( \triangle ABC \) be a rectangular triangle. The equality \( a = \frac{b + c}{2} \) holds in \( \triangle ABC \) iff \( p = 6r, R = \frac{5}{2}r \) (i.e. in this case, \( \triangle ABC \) is homothetic to a triangle with sides 3, 4, 5).

(2) Let \( \triangle ABC \) be a triangle with angle 60°. The equality \( a = \frac{b + c}{2} \) holds in \( \triangle ABC \) iff \( \triangle ABC \) is an equilateral triangle.

(3) Let \( \triangle ABC \) be a triangle with angle 120°. The equality \( a = \frac{b + c}{2} \) holds in \( \triangle ABC \) iff \( p = 5\sqrt{3}r, R = \frac{14}{3}r \) (i.e. in this case, \( \triangle ABC \) is homothetic to a triangle with sides 6, 10, 14).

2. Proof of the main results.

Proof of Theorem 1. Let \( \triangle ABC \) be an arbitrary triangle such that \( a = \frac{b + c}{2} \). Then \( a = \frac{2p}{3} \).

By [8], \( a, b, c \) are roots of the equation
\[ x^3 - 2px^2 + (p^2 + r^2 + 4Rr)x - 4pRr = 0. \] (1)

Therefore
\[ \left( \frac{2p}{3} \right)^3 - 2p \left( \frac{2p}{3} \right)^2 + (p^2 + r^2 + 4Rr) \cdot \frac{2p}{3} - 4pRr = 0 \]

and \( p^2 = 18Rr - 9r^2 \). So (1) implies (2).

Now prove that (2) implies (1). Assume that \( p^2 = 18Rr - 9r^2 \). Then \( \frac{2p}{3} \) is a root of the equation (1) and
\[ x^3 - 2px^2 + (p^2 + r^2 + 4Rr)x - 4pRr = \left( x - \frac{2p}{3} \right) \left( x^2 - \frac{4p}{3}x + 6Rr \right). \]

It implies that the numbers
\[ b = \frac{2p}{3} - \sqrt{2r(R - 2r)}, a = \frac{2p}{3}, \]
\[ c = \frac{2p}{3} + \sqrt{2r(R - 2r)} \]

are roots of (1) and the equality \( a = \frac{2p}{3} = \frac{b + c}{2} \) holds in \( \triangle ABC \). Note that we have proved also that the condition (1) is equivalent to the condition (3).

We will show \( \alpha = \angle BAC \leq 60^\circ \). We need to prove the following lemma.

Lemma. \( \cos \alpha = 1 - \frac{r}{R} \).

Proof. The equality \( a = \frac{b + c}{2} \) is equivalent to \( a^2 = (b + c - a)^2 \). In its turn, the condition \( a^2 = (b + c - a)^2 \) is equivalent to the equality
\[ 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2(1 - \frac{a^2 + c^2 - b^2}{2ac})}{1 - \frac{a^2 + b^2 - c^2}{2ab}} \]

and hence
\[ 1 - \cos \alpha = \frac{4}{2} \left( 1 - \frac{\sin ^2 \alpha}{2} \right) = \frac{1 - \cos \gamma}{2}. \]

The last equality can be written as \( \sin ^2 \alpha = \frac{4\sin ^2 \beta \sin ^2 \gamma}{2} \), i.e. \( \sin \alpha = 2\sin \beta \sin \gamma \). By [8],
\[ \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{r}{2R}. \]

Thus
\[ \sin ^2 \frac{\alpha}{2} = 2\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{r^2}{2R} \]

Hence \( \cos \alpha = 1 - 2\sin ^2 \frac{\alpha}{2} = 1 - \frac{r}{R} \) and the lemma is proved.

It is known that \( R \geq 2r \) (see [8–10]). Therefore \( \cos \alpha = 1 - \frac{r}{R} \geq 1 - \frac{1}{2} \) and \( \alpha \leq 60^\circ \). The proof is complete.

Proof of Theorem 2. Assume that \( R, r \) are arbitrary positive numbers such that \( R \geq 2r \) and \( p = \sqrt{18Rr - 9r^2} \). By [8, Theorem 2, P. 54], positive numbers \( R, r, p \) are the circumradius, the inradius, the semiperimeter of some triangle respectively iff the condition
\[ (p^2 - 2R^2 - 10Rr + r^2)^2 \leq 4R(R - 2r)^3 \] (2) holds. Note that \( p = \sqrt{18Rr - 9r^2} \) implies (2).

Really, we have
\[ (p^2 - 2R^2 - 10Rr + r^2)^2 = 4(R - 2r)^4 \leq 4R(R - 2r)^3. \]

The proof is complete.

Proof of Theorem 3. Prove (1). Consider a rectangular triangle \( \triangle ABC \) in which \( \gamma = 90^\circ \). By [8, P. 26], \( \cos \alpha \cos \beta \cos \gamma = \frac{1}{4R^2}(p^2 - (2R + r)^2) \).

Since \( \gamma = \angle ACB = 90^\circ \), we have \( p = 2R + r \). By Theorem 1, the equality \( a = \frac{b + c}{2} \) is equivalent to the condition \( p^2 = 18Rr - 9r^2 = (2R + r)^2 \), or \( 2R^2 - 7Rr + 5r^2 = (R - r)(2R - 5r) = 0 \). Since \( R \geq 2r \)
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(the Euler's inequality), the equality \( a = \frac{b + c}{2} \) is equivalent to \( R = \frac{5r}{2} \) for the rectangular triangle \( \triangle ABC \). So \( p = 2R + r = 6r, a = \frac{2p}{3} = 4r, c = 2R = 5r, b = 3r \) for such triangle. Thus \( \triangle ABC \) is homothetic to a triangle with sides 3, 4, 5.

Further prove (2). Let \( \triangle ABC \) be a triangle with angle 60°. By [8], the numbers \( \cos \alpha, \cos \beta, \cos \gamma \) are roots of the equation

\[
4R^2x^3 - 4R(R + r)x^2 + (p^2 + r^2 - 4R^2)x + (2R + r)^2 - p^2 = 0. 
\]

Hence \( \cos 60° = \frac{1}{2} \) is a root of (3) and \( p = \sqrt{3}(R + r) \).

By Theorem 1, the equality \( a = \frac{b + c}{2} \) is equivalent to the condition \( p^2 = 18Rr - 9r^2 \). Since \( p^2 = 3(R^2 + 2Rr + r^2) \), we have \( 3R^2 + 6Rr + 3r^2 = 18Rr - 9r^2 \), i.e. \( (R - 2r)^2 = 0 \). Therefore the equality \( a = \frac{b + c}{2} \) is equivalent to \( R = 2r \) for any triangle with angle 60°. By [8, P. 8], the last inequality holds if and only if \( \triangle ABC \) is an equilateral triangle.

Finally, prove (3). Let \( \triangle ABC \) be a triangle with angle 120°. Then \( \cos 120° = -\frac{1}{2} \) is a root of (3) and \( p = \frac{3R + r}{\sqrt{3}} \). Thus, by Theorem 1, the equality \( a = \frac{b + c}{2} \) is equivalent to the condition \( p^2 = 18Rr - 9r^2 \). Since \( p^2 = 3(R^2 + 2Rr + r^2) \), we have \( 3R^2 + 6Rr + 3r^2 = 18Rr - 9r^2 \), i.e. \( (R - 2r)^2 = 0 \). Therefore the equality \( a = \frac{b + c}{2} \) is equivalent to \( R = 2r \) for any triangle with angle 60°. By [8, P. 8], the last inequality holds if and only if \( \triangle ABC \) is an equilateral triangle.

Case 1. Let \( R = (4 + 2\sqrt{3})r \).

In this case, by Theorem 1, the sides of the triangle \( ABC \) are equal to

\[
a = \frac{2p}{3} = \frac{2}{3}(R + r(2 + \sqrt{3})) = \frac{2}{3}(6 + 3\sqrt{3})r = (4 + 2\sqrt{3})r, \\
b = \frac{2p}{3} - \sqrt{2r(R - 2r)} = \frac{2}{3}(R + r(2 + \sqrt{3})) - \sqrt{2r(2 + 2\sqrt{3})r} = \left(4 + 2\sqrt{3} - 2\sqrt{\sqrt{3} + 1}\right)r, \\
c = \frac{2p}{3} + \sqrt{2r(R - 2r)} = \left(4 + 2\sqrt{3} + 2\sqrt{\sqrt{3} + 1}\right)r 
\]

and \( \triangle ABC \) is homothetic to a triangle with sides

\[
\left(4 + 2\sqrt{3} + 2\sqrt{\sqrt{3} + 1}\right), 4 + 2\sqrt{3}, \\
\left(4 + 2\sqrt{3} - 2\sqrt{\sqrt{3} + 1}\right). 
\]

Case 2. Let \( R = (10 - 4\sqrt{3})r \).

In this case, the sides and the semiperimeter of the triangle \( ABC \) are equal to

\[
a = \frac{2p}{3} = \frac{2}{3}(12 - 3\sqrt{3}) = (8 - 2\sqrt{3})r, \\
p = R + r(2 + \sqrt{3}) = (10 - 3\sqrt{3})r, \\
b = \frac{2p}{3} - \sqrt{2r(R - 2r)} = (10 - 4\sqrt{3})r, \\
c = \frac{2p}{3} + \sqrt{2r(R - 2r)} = 6r. 
\]

Then \( \triangle ABC \) is homothetic to a triangle with sides

\[
8 - 2\sqrt{3}, 10 - 4\sqrt{3}, 6. 
\]

So the following theorem is true.

**Theorem 4.** Let \( ABC \) be a triangle with angle 30° and the equality \( a = \frac{b + c}{2} \) holds in \( \triangle ABC \). Then either \( \triangle ABC \) is homothetic to a triangle with sides

\[
8 - 2\sqrt{3}, 10 - 4\sqrt{3}, 6, 
\]

or \( \triangle ABC \) is homothetic to a triangle with sides

\[
\left(4 + 2\sqrt{3} + 2\sqrt{\sqrt{3} + 1}\right), 4 + 2\sqrt{3}, \\
\left(4 + 2\sqrt{3} - 2\sqrt{\sqrt{3} + 1}\right). 
\]
Consider the class $K(\varphi)$ of triangles with fixed angle $\varphi (0 < \varphi < \pi)$ such that the equality $a = \frac{b + c}{2}$ holds for sides $a, b, c$ (in particular, if $b, a, c$ form an arithmetic progression). By [8, p. 26],

$$p^2(1 - \cos \varphi) = 4R^2 \cos^3 \varphi - 4R(R + r) \cos^2 \varphi + (r^2 - 4R^2) \cos \varphi + (2R + r)^2 =$$

$$= R^2(4\cos^3 \varphi - 4 \cos^2 \varphi - 4 \cos \varphi + 4) + 4Rr \cdot (1 - \cos^2 \varphi) + r^2(1 + \cos \varphi),$$

$$p^2 = R^2 \cdot 4\sin^2 \varphi + 4(1 + \cos \varphi)Rr + \frac{ctg^2 \varphi}{2} r^2 = 18Rr - 9r^2.$$ 

Let $\lambda = \frac{R}{r} \geq 2$. Then $\lambda$ is a root of

$$4\sin^2 \varphi \cdot t^2 + 4(\cos \varphi - 14)t + (ctg^2 \varphi + 9) = 0. \quad (4)$$

The equation $(4)$ has no more than two different roots $\lambda_1, \lambda_2$. Let $r$ be a fixed positive number. For each $\lambda_i (i = 1, 2)$, we can calculate $R_i = \lambda_ir,$ $p_i = \sqrt{18R_i r - 9r^2} = \sqrt{18\lambda_i - 9},$ $a_i = \frac{2p_i}{3},$ $b_i = \frac{2p_i}{3} - \sqrt{2r(R_i - 2r)},$ $c_i = \frac{2p_i}{3} + \sqrt{2r(R_i - 2r)}, i \leq 2.$

So the class $K(\varphi)$ has no more than two different subclasses of homothetic triangles. For example, we have:

1. the class $K(90^\circ)$ consists of triangles that are homothetic to a triangle with sides 3, 4, 5;
2. the class $K(30^\circ)$ consists of triangles that are homothetic to a triangle with sides either $8 - 2\sqrt{3}, 6, 10 - 4\sqrt{3},$ or $(4 + 2\sqrt{3} + 2\sqrt{3} + 1), 4 + 2\sqrt{3}, (4 + 2\sqrt{3} - 2\sqrt{3} + 1);$
3. the class $K(60^\circ)$ consists of equilateral triangles;
4. the class $K(120^\circ)$ consists of triangles that are homothetic to a triangle with sides 3, 5, 7.

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